KCC-invariants-based geometrization of a theory of electromagnetic and spinor fields on the background of the Schwarzschild spacetime

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Introduction

The Kosambi–Cartan–Chern geometrical approach (KCC-theory) is developed in detail in numerous mathematical books and papers [1–3]. KCC-theory allows to describe the evolution of a dynamical system in a configuration space of Lagrange type. At that, the dynamical system is given by the system of second-order differentials equations:

$$\frac{dy^i}{dr} + 2G^i = 0, \qquad y^i = \frac{dx^i}{dr}$$

$$G^i = \frac{1}{4} g^{il} \left(\frac{\partial^2 L}{\partial x^k \partial y^l} y^k - \frac{\partial L}{\partial x^l} + \frac{\partial^2 L}{\partial y^l \partial t} \right), \qquad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$$

and its properties are described in terms of five KCC geometrical invariants. From the physical point of view, the most interesting invariant is the second one which is associated with the Jacobi stability of the system, i.e. convergence or divergence of the bundles of geodesics.

$$P^i_j = 2\frac{\partial G^i}{\partial x^j} + 2G^s \frac{\partial N^i_j}{\partial y^s} - \frac{\partial N^i_j}{\partial x^s} y^s - N^i_s N^s_j - \frac{\partial N^i_j}{\partial r}, \qquad N^i_j = \frac{\partial G^i}{\partial y^j}$$

In this work we apply the KCC-theory to study systems of differential equations which arise in theory of electro-magnetic and spinor fields on the background of the Schwarzschild spacetime.

RESULTS AND DISCUSSION

The electromagnetic field on the background of the curved space-time of a Schwarzschild black hole has been considered. The first and second KCC-invariants for the second-order differential equations obtained after variable splitting in initial Maxwell equations both in the complex Majorana–Oppenheimer formalism and 10-dimensional Duffin–Kemmer–Petiau formalism. The second KCC-invariant determines Jacobi field for geodesics deviation, so it indicates how rapidly the different branches of the solution diverge from or converge to the intersection points.

From physical point of view, the most interesting points are the singular points. The numerical calculated dependencies of the second invariant on the radial variable are shown in Figure 1a. Near the singular points the second invariant Λ behaves as follows:

$$x = \frac{r}{M} \to 0 \qquad \Lambda \to -\frac{3}{4x^2} < 0;$$

$$x \to 1 \; (r \to r_g \equiv M) \qquad \Lambda \to \frac{1 + 4M^2\omega^2}{4(x-1)^2} > 0;$$

$$x \to \infty \; (r \to \infty) \qquad \Lambda \to \omega^2 > 0.$$

It indicates that in the vicinity of x=0 the geodesics converge. Vice versa, near the Schwarzschild horizon x=1 and at x→∞ the Jacobi instability exists and the geodesics diverge.

For a spin ½ particle on the background of Schwarzschild spacetime the radial differential equation system obtained from Dirac equation after variable splitting has been studied. The first and second KCC-invariants are determined explicitly. The behavior of two different eigenvalues Λ_1,2 of the second invariant has been analyzed numerically (see Figure 1b). In the vicinity of the singular points the real parts of the eigenvalues behave as follows:

$$r \to 0 \qquad \Lambda_{1,2} \to -\frac{5}{16r^2} < 0;$$

$$r \to M \qquad Re[\Lambda_{1,2}] \to \frac{r^2\varepsilon^2 + 3}{4(M-r)^2} > 0;$$

$$r \to \infty \qquad \Lambda_{1,2} \to \frac{1}{4}(\varepsilon^2 - m^2).$$

In the vicinity of the Schwarzschild horizon the geodesics diverge at any energy ε, while at r→∞ the geodesics diverge at ε>m and converge at ε<m. The 3-d, 4-th and 5-th invariants for the both considered problems are equal to zero.

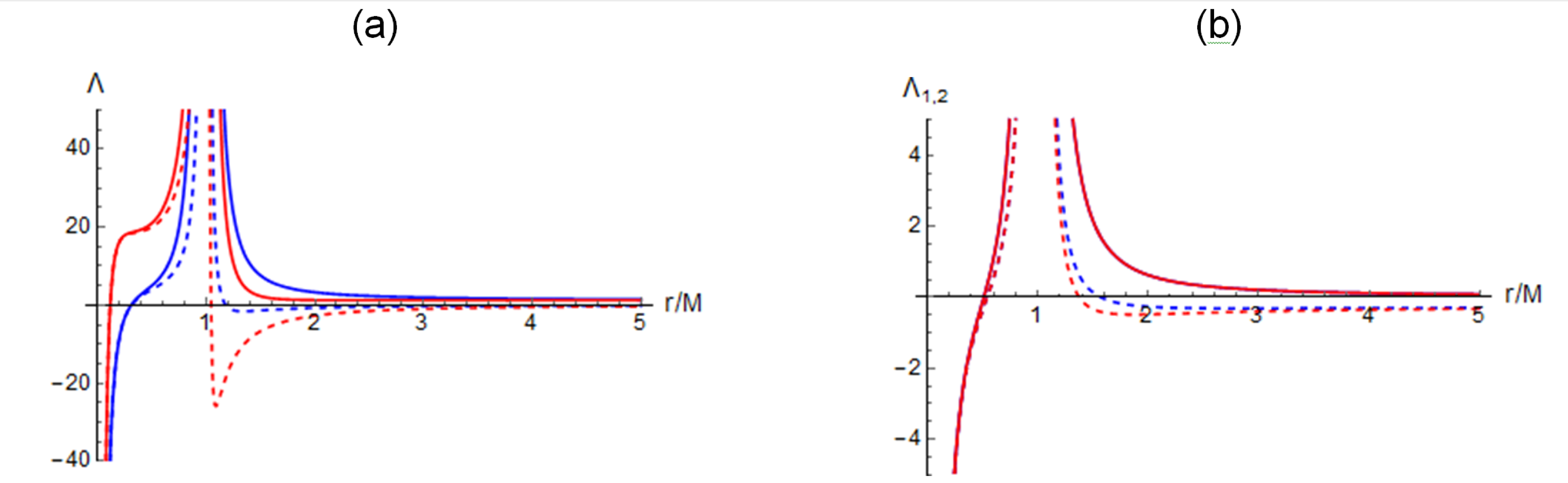

Figure 1. Typical dependencies of the second invariant eigenvalues on the radial coordinate $x = \frac{r}{M}$ for the geometrized problem of the electromagnetic (a) and spinor (b) fields on the background of the Schwarzschild spacetime. The values of parameters used are: (a) M = 1, ω = 0.0001 (dashed curves) and 1.0001 (solid curves), j = 1 (blue) and 2 (red); (b) m = 1, M = 1, ε = 0.0001 (dashed curves) and 1.0001 (solid curves), j = ½; blue and red curves correspond to two different eigenvalues Λ_1 and Λ_2.

Conclussions

So, we apply the KCC-geometrical approach to study the radial equation systems arising in two quantum-mechanical problems, i.e. electromagnetic and spinor fields on the background of the Schwarzschild spacetime. The stability analysis in terms of the second invariant demonstrate the difference in the behavior of geodesics at r→∞ for these two problems that may be associated with different structure of solution (discrete and continuous spectra). The vanishing of the 3-d, 4-th and 5-th invariants means that, in geometrical terms, there exists a nonlinear connection on the tangent bundle, with zero torsion and curvature.

References

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